Prove the correctness of this exponentiation algorithm:
$x \geq 0$ { (set z 1) (set y x) (while (> y 0) (begin (set z (* z 2)) (set y (- y 1)))) } $z = 2^x$

Hint: Loop invariant is $y \geq 0 \wedge z = 2^{x-y}$

Note that there is an implicit "begin" around the entire code fragment. So the given code is equivalent to:
(begin (set z 1) (set y x) (while (> y 0) (begin (set z (* z 2)) (set y (- y 1)))))

Here is a complete proof:

Step	Precondition	Code fragment or implication	Postcondition	Reason
1	$x \geq 0$	$\rightarrow$	$1 = 2^{x-x} \wedge x \geq 0$	axiom
2	$1 = 2^{x-x} \wedge x \geq 0$	(set z 1)	$z = 2^{x-x} \wedge x \geq 0$	R0
3	$x \geq 0$	(set z 1)	$z = 2^{x-x} \wedge x \geq 0$	R1; 1,2
4	$z = 2^{x-x} \wedge x \geq 0$	(set y x)	$z = 2^{x-y} \wedge y \geq 0$	R0
5	$x \geq 0$	(set z 1) (set y x)	$z = 2^{x-y} \wedge y \geq 0$	R2; 3,4
6	$z = 2^{x-y} \wedge y \geq 0 \wedge y > 0$	$\rightarrow$	$z*2 = 2^{x-(y-1)} \wedge y-1 \geq 0$	axiom
7	$z*2 = 2^{x-(y-1)} \wedge y-1 \geq 0$	(set z (* z 2))	$z = 2^{x-(y-1)} \wedge y-1 \geq 0$	R0
8	$z = 2^{x-y} \wedge y \geq 0 \wedge y > 0$	(set z (* z 2))	$z = 2^{x-(y-1)} \wedge y-1 \geq 0$	R1; 6,7
9	$z = 2^{x-(y-1)} \wedge y-1 \geq 0$	(set y (- y 1))	$z = 2^{x-y} \wedge y \geq 0$	R0
10	$z = 2^{x-y} \wedge y \geq 0 \wedge y > 0$	(begin (set z (* z 2)) (set y (- y 1))	$z = 2^{x-y} \wedge y \geq 0$	R2; 8,9
11	$z = 2^{x-y} \wedge y \geq 0$	(while (> y 0) (begin (set z (* z 2)) (set y (- y 1))))	$z = 2^{x-y} \wedge y \geq 0 \wedge \neg y > 0$	R4; 10
12	$z = 2^{x-y} \wedge y \geq 0 \wedge \neg y > 0$	$\rightarrow$	$z = 2^x$	axiom
13	$z = 2^{x-y} \wedge y \geq 0$	(while (> y 0) (begin (set z (* z 2)) (set y (- y 1))))	$z = 2^x$	R1; 11,12
14	$x \geq 0$	(set z 1) (set y x) (while (> y 0) (begin (set z (* z 2)) (set y (- y 1))))	$z = 2^x$	R2; 5,13

Prove the correctness of this logarithm algorithm:
$x \geq 1 \wedge b \geq 2$ { (set j x) (set k 0) (while (> j b) (begin (set j (/ j b)) (set k (+ k 1)))) } $k = \lfloor \log_b x \rfloor$

Hint: Loop invariant is $j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b(x/j) \rfloor$

Here is a complete proof:

Step	Precondition	Code fragment or implication	Postcondition	Reason
1	$x \geq 1 \wedge b \geq 2$	$\rightarrow$	$x \geq 1 \wedge b \geq 2 \wedge x/x = 1$	axiom
2	$x \geq 1 \wedge b \geq 2 \wedge x/x = 1$	(set j x)	$j \geq 1 \wedge b \geq 2 \wedge x/j = 1$	R0
3	$x \geq 1 \wedge b \geq 2$	(set j x)	$j \geq 1 \wedge b \geq 2 \wedge x/j = 1$	R1; 1,2
4	$j \geq 1 \wedge b \geq 2 \wedge x/j = 1$	$\rightarrow$	$j \geq 1 \wedge b \geq 2 \wedge 0 = \lfloor \log_b x/j \rfloor$	axiom
5	$j \geq 1 \wedge b \geq 2 \wedge 0 = \lfloor \log_b x/j \rfloor$	(set k 0)	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor$	R0
6	$j \geq 1 \wedge b \geq 2 \wedge x/j = 1$	(set k 0)	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor$	R1; 4,5
7	$x \geq 1 \wedge b \geq 2$	(set j x) (set k 0)	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \lfloor \log_b x/j \rfloor$	R2; 3,6
8	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge j \geq b$	(set j (/ j b))	$bj \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor - 1 \wedge bj \geq b$	R0
9	$bj \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor - 1 \wedge bj \geq b$	(set k (+ k 1))	$bj \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge bj \geq b$	R0
10	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge j \geq b$	(begin(set j (/ j b)) (set k (+ k 1)))	$bj \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge bj \geq b$	R2; 8,9
11	$bj \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge bj \geq b$	$\rightarrow$	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor$	axiom
12	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge j \geq b$	(begin(set j (/ j b)) (set k (+ k 1)))	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor$	R1; 10,11
13	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor$	(while (> j b) (begin (set j (/ j b)) (set k (+ k 1))))	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge \neg j \geq b$	R4; 12
14	$j \geq 1 \wedge b \geq 2 \wedge k = \lfloor \log_b x/j \rfloor \wedge \neg j \geq b$	$\rightarrow$	$k > \lfloor \log_b x/b \rfloor = \lfloor \log_b x \rfloor$	axiom
15	$x \geq 1 \wedge b \geq 2$	(set j x) (set k 0) (while (> j b) (begin (set j (/ j b)) (set k (+ k 1))))	$k = \lfloor \log_b x \rfloor$	R2; 7,13,14

Prove the correctness of the Russian Peasants' multiplication algorithm:
$x \geq 0$ { (set n 0) (set a x) (set b y) (while (> a 0) (begin (if (= (mod a 2) 1) (set n (+ n b)) 0) (set a (/ a 2)) (set b (* b 1)))) } $n = x \cdot y$

Hint: Loop invariant is $a \geq 0 \wedge n = x \cdot y - a \cdot b$

Here is a complete proof:

Step	Precondition	Code fragment or implication	Postcondition	Reason
1	$x \geq 0$	(set n 0)	$x \geq 0 \wedge n = 0$	R0
2	$x \geq 0 \wedge n = 0$	(set a x)	$a \geq 0 \wedge n = 0 \wedge x = a$	R0, axiom
3	$a \geq 0 \wedge n = 0 \wedge x = a$	(set b y)	$a \geq 0 \wedge n = 0 \wedge x = a \wedge y = b$	R0
4	$a \geq 0 \wedge n = 0 \wedge x = a \wedge y = b$	$\rightarrow$	$a \geq 0 \wedge n = x \cdot y - a \cdot b$	axiom
5	$a \geq 0 \wedge n = 0 \wedge x = a$	(set b y)	$a \geq 0 \wedge n = x \cdot y - a \cdot b$	R1; 3,4
6	$x \geq 0$	(set n 0) (set a x) (set b y)	$a \geq 0 \wedge n = x \cdot y - a \cdot b$	R2; 1,2,5
7	$a \geq 0 \wedge n = x \cdot y – a \cdot b \wedge a > 0 \wedge a \bmod 2 = 1$	(set n (+ n b))	$a \geq 0 \wedge n - b = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 1$	R0
8	$a \geq 0 \wedge n - b = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 1$	$\rightarrow$	$a \geq 0 \wedge a > 0 \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b$	axiom
9	$a \geq 0 \wedge n = x \cdot y – a \cdot b \wedge a > 0 \wedge a \bmod 2 = 1$	(set n (+ n b))	$a \geq 0 \wedge a > 0 \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b$	R1; 7,8
10	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 0$	0	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 0$	R0
11	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 0$	$\rightarrow$	$a \geq 0 \wedge a > 0 \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b$	axiom
12	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0 \wedge a \bmod 2 = 0$	0	$a \geq 0 \wedge a > 0 \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b$	R1; 10,11
13	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0$	(if (= (mod a 2) 1) (set n (+ n b)) 0)	$a \geq 0 \wedge a > 0 \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b$	R3; 9,12
14	$a \geq 0 \ \wedge n = x \cdot y - a \cdot b + (a \bmod 2)\ b \wedge$	(set a (/ a 2))	$a \geq 0 \ \wedge n = x \cdot y - 2a \cdot b \wedge a > 0$	R0, axiom

	$a > 0$			
15	$a \geq 0 \ \wedge n = x \cdot y - 2a \cdot b \wedge a > 0$	(set b (* b 1))	$a \geq 0 \ \wedge n = x \cdot y - a \cdot b \wedge a > 0$	R0
16	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge a > 0$	(begin (if (= (mod a 2) 1) (set n (+ n b)) 0) (set a (/ a 2)) (set b (* b 1)))	$a \geq 0 \ \wedge n = x \cdot y - a \cdot b \wedge a > 0$	R2; 13,14,15
17	$a \geq 0 \wedge n = x \cdot y - a \cdot b$	(while (> a 0) (begin (if (= (mod a 2) 1) (set n (+ n b)) 0) (set a (/ a 2)) (set b (* b 1))))	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge \neg a > 0$	R4; 16
18	$x \geq 0$	(set n 0) (set a x) (set b y) (while (> a 0) (begin (if (= (mod a 2) 1) (set n (+ n b)) 0) (set a (/ a 2)) (set b (* b 1))))	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge \neg a > 0$	R2; 6, 17
19	$a \geq 0 \wedge n = x \cdot y - a \cdot b \wedge \neg a > 0$	$\rightarrow$	$n = x \cdot y$	axiom
20	$x \geq 0$	(set n 0) (set a x) (set b y) (while (> a 0) (begin (if (= (mod a 2) 1) (set n (+ n b)) 0) (set a (/ a 2)) (set b (* b 1))))	$n = x \cdot y$	R1; 18, 19